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Convolution formulae for functions of Rayleigh type

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Abstract. N Kishore (1963 *Proc. Am. Math. Soc.* **14** 527) considered the Rayleigh functions $\sigma_n(\nu) = \sum_{k=1}^{\infty} j_{\nu k}^{-2n}$, $n = 1, 2, \ldots$, where the $j_{\nu k}$ are the (non-zero) zeros of the Bessel function $J_\nu(z)$ and provided a convolution-type sum formula for finding σ_n in terms of $\sigma_1, \ldots, \sigma_{n-1}$. Here we investigate corresponding expressions for sums of reciprocal powers of zeros τ_n of derivatives and other functions related to Bessel functions. It turns out that we can get results similar to Kishore's expressing τ_n in terms of $\tau_1, \ldots, \tau_{n-1}$ *and* $\sigma_1, \ldots, \sigma_n$.

An old method due to Euler, Rayleigh and others for evaluating the zeros $\pm j_{\nu k}$ of the Bessel function

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+\nu}}{n!\Gamma(\nu+n+1)} \tag{1}$$

is based on the Rayleigh functions defined [1, p 502] by the formula

$$\sigma_n(\nu) = \sum_{k=1}^{\infty} j_{\nu k}^{-2n} \qquad n = 1, 2, \ldots. \tag{2}$$

For example, the inequalities

$$[\sigma_n(\nu)]^{-1/n} < j_{\nu 1}^2 < \sigma_n(\nu)/\sigma_{n+1}(\nu) \qquad \nu > -1,\ n = 1, 2, \ldots \tag{3}$$

provide infinite sequences of successively improving upper and lower bounds for $j_{\nu 1}^2$. Hence, it is important to find 'sum rules' or formulae for $\sigma_n(\nu)$. The method originating with Euler (see [1, p 500] for details) uses a generalization to entire functions of Newton's formula for sums of powers of roots of a polynomial in terms of its coefficients; various ramifications of the method were considered recently in [2]. By this method, we can find all the $\sigma_n(\nu)$ in terms of the coefficients in the series (1).

The fact that the Bessel function satisfies additional relations including the differential equation

$$z^2 y'' + z y' + (z^2 - \nu^2) y = 0 \tag{4}$$

and the recurrence formula

$$z J_\nu'(z) - \nu J_\nu(z) = -z J_{\nu+1}(z) \tag{5}$$

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suggests that there may be other approaches to finding the $\sigma_n(\nu)$. Kishore [3] has provided a compact convolution formula

$$\sigma_n(\nu) = \frac{1}{\nu+n}\sum_{k=1}^{n-1}\sigma_k(\nu)\sigma_{n-k}(\nu) \tag{6}$$

from which the $\sigma_n(\nu)$ may be found successively, starting from

$$\sigma_1 = 1/[4(\nu+1)]. \tag{7}$$

The next two sums in order are

$$\sigma_2(\nu) = \frac{1}{16(\nu+1)^2(\nu+2)} \qquad \sigma_3(\nu) = \frac{1}{32(\nu+1)^3(\nu+2)(\nu+3)}.$$

Formula (6) is useful in proving higher monotonicity properties of the Rayleigh functions [4] and in deriving congruence properties for some of their functional values [5].

The question arises whether there are analogues of the Kishore formula (6) for the zeros of other special functions such as the first and second derivatives of the Bessel function. Here we give a variant of this result for the zeros of the more general function

$$N_\nu(z) = az^2J_\nu''(z) + bzJ_\nu'(z) + cJ_\nu(z) \tag{8}$$

considered by Mercer [6]. Using (4), we have

$$N_\nu(z) = (a\nu^2 - az^2 + c)J_\nu(z) + (b-a)zJ_\nu'(z) \tag{9}$$

and

$$zN_\nu'(z) = [-2az^2 + (b-a)(\nu^2 - z^2)]J_\nu(z) + (a\nu^2 - az^2 + c)zJ_\nu'(z). \tag{10}$$

It is convenient to consider the function

$$y_\nu(z) = N_\nu(z^{1/2}) = \frac{a\nu^2 + c + (b-a)\nu}{2^\nu\Gamma(\nu+1)} z^{\nu/2}\prod_{k=1}^{\infty}\left(1 - \frac{z}{\zeta_k}\right) \tag{11}$$

where we choose that branch of $z^{1/2}$ which is positive for $z > 0$ and the ζ_k are the zeros of $y_\nu(z)$ or the squares of the zeros of the even entire function $z^{-\nu}N_\nu(z)$. The constant multiplicative factor is obtained from the series (1). The validity of this infinite product expansion follows from facts on entire functions of finite order [7, chapter 8]. We may differentiate (11) logarithmically [8], to obtain

$$\frac{y_\nu'(z)}{y_\nu(z)} = \frac{\nu}{2z} - \sum_{k=1}^{\infty}\frac{1/\zeta_k}{1 - z/\zeta_k} = \frac{\nu}{2z} - \sum_{k=1}^{\infty}\frac{1}{\zeta_k}\sum_{n=0}^{\infty}\frac{z^n}{\zeta_k^n}.$$

This gives

$$2z\frac{y_\nu'(z)}{y_\nu(z)} = \frac{z^{1/2}N_\nu'(z^{1/2})}{N_\nu(z^{1/2})} = \nu - 2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} z^n/\zeta_k^n.$$

But we may interchange the orders of summation here (since the iterated series converges absolutely) to obtain

$$\frac{z^{1/2}N_\nu'(z^{1/2})}{N_\nu(z^{1/2})} = \nu - 2\sum_{n=1}^{\infty} z^n \sum_{k=1}^{\infty}\zeta_k^{-n} = \nu - 2\sum_{n=1}^{\infty}\tau_n z^n \tag{12}$$

where

$$\tau_n = \sum_{k=1}^{\infty}\zeta_k^{-n}. \tag{13}$$

In particular, we have as in [3],

$$\frac{z^{1/2}J'_\nu(z^{1/2})}{J_\nu(z^{1/2})} = \nu - 2\sum_{n=1}^{\infty}\sigma_n z^n. \tag{14}$$

Using (9) and (10), we can write (12) as

$$\left[\nu - 2\sum_{n=1}^{\infty}\tau_n z^n\right] \times \left[A(\nu) - az - 2q\sum_{n=1}^{\infty}\sigma_n z^n\right]$$
$$= q\nu^2 - (a+b)z + (a\nu^2 - az + c)\left[\nu - 2\sum_{n=1}^{\infty}\sigma_n z^n\right] \tag{15}$$

where we have used the abbreviated notations

$$A(\nu) = a\nu^2 + (b-a)\nu + c \qquad q = b - a. \tag{16}$$

Comparing coefficients of powers of z^n, $n = 1, 2, \ldots$ on both sides of (15), we find

$$2A(\nu)\tau_1 + 2q\nu\sigma_1 = a + b + 2(a\nu^2 + c)\sigma_1 \tag{17}$$

and

$$A(\nu)\tau_n + \nu q\sigma_n - a\tau_{n-1} - 2q\sum_{k=1}^{n-1}\sigma_k\tau_{n-k} = -a\sigma_{n-1} + (a\nu^2 + c)\sigma_n \qquad n = 2, 3, \ldots. \tag{18}$$

This leads to

$$\tau_1 = \frac{a + b + 2[a\nu^2 + (a-b)\nu + c]\sigma_1}{2A(\nu)}$$

or recalling (7),

$$\tau_1 = \frac{a\nu^2 + (3a+b)\nu + 2(a+b) + c}{4(\nu+1)[a\nu^2 + (b-a)\nu + c]} \tag{19}$$

and

$$A(\nu)\tau_n = (a\nu^2 + c - \nu q)\sigma_n - a(\sigma_{n-1} - \tau_{n-1}) + 2q\sum_{k=1}^{n-1}\sigma_k\tau_{n-k} \qquad n = 2, 3 \ldots. \tag{20}$$

In the special case $a = b = 0$, $c = 1$, where we are dealing with the zeros of the Bessel function, these reduce, as they should, to

$$\tau_1 = \frac{1}{4(\nu+1)} \qquad \tau_n = \sigma_n.$$

In the special case $a = c = 0$, $b = 1$, $q = 1$, $A(\nu) = \nu$, we are dealing with the non-trivial zeros of the function $J'_\nu(z)$; (19) and (20) become

$$\tau_1 = \sum_{k=1}^{\infty}[j'_{\nu k}]^{-2} = \frac{\nu+2}{4\nu(\nu+1)} \tag{21}$$

and

$$\nu\tau_n = \nu\sum_{k=1}^{\infty}[j'_{\nu k}]^{-2n} = -\nu\sigma_n + 2\sum_{k=1}^{n-1}\sigma_k\tau_{n-k} \qquad n = 2, 3 \ldots. \tag{22}$$

In particular, these lead to

$$\tau_2 = \sum_{k=1}^{\infty} [j'_{\nu k}]^{-4} = \frac{1}{16} \frac{\nu^2 + 8\nu + 8}{\nu^2(\nu+1)^2(\nu+2)} \tag{23}$$

$$\tau_3 = \sum_{k=1}^{\infty} [j'_{\nu k}]^{-6} = \frac{1}{32} \frac{\nu^3 + 16\nu^2 + 38\nu + 24}{\nu^3(\nu+1)^3(\nu+2)(\nu+3)}. \tag{24}$$

These are the same results as obtained by the power series method in [2].

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